# Dynamics of One-Dimensional Integrable Systems

PhD Thesis Booklet

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2024.

# Introduction

One-dimensional integrable quantum systems represent a special class of many-body models characterized by the presence of an infinite number of conserved quantities. These conserved quantities strictly constrain their dynamics and allow for exact analytical solutions, such as those obtained through the Bethe Ansatz, making integrable systems solvable in ways that are impossible in generic models. This solvability provides deep insights into phenomena that would otherwise be intractable or would require difficult numerical methods in generic many-body systems. As a result, integrable models can serve as benchmarks, offering clear examples of quantum behavior that enhance our understanding of more intricate and less tractable systems.

In recent years, the study of integrable systems has seen a surge in interest, largely driven by experimental developments. Advances in cold atom experiments and other quantum simulation techniques have made it possible to realize nearly perfect integrable systems in laboratory settings, thus allowing for direct observation of their unique dynamical properties. In particular, the study of quantum quenches, where an integrable system is taken out of equilibrium and allowed to evolve, has become a major focus, leading to a deeper understanding of how such systems relax and evolve over time.

One of the most striking features observed in such experiments is that integrable systems do not thermalize in the conventional sense. While generic systems are expected to relax to a thermal equilibrium described by the Gibbs ensemble, integrable systems, due to their conserved charges, avoid this process. Instead, they equilibrate to a state described by the Generalized Gibbs Ensemble (GGE), which accounts for the additional conserved quantities. This framework has become essential for explaining the long-time behavior of integrable systems after a quantum quench, showing how these systems retain information about their initial conditions.

Beyond equilibration, the transport properties of integrable systems have also been the subject of intense investigation. The discovery of Generalized Hydrodynamics (GHD) in the past decade has revolutionized the understanding of how conserved quantities, such as energy and particle density, propagate in integrable models. GHD provides a hydrodynamic description that has proven to be particularly useful for describing non-equilibrium steady states in inhomogeneous systems and has opened new possibilities for studying transport phenomena in integrable models.

The thesis makes several contributions to the field of out-of-equilibrium dynamics of integrable quantum systems, particularly by investigating some foundational assumptions of GHD. Through rigorous analysis of the current operators of integrable models, it provides a deeper understanding of GHD and its theoretical underpinnings. Additionally, the thesis explores systems that exhibit a different form of thermalization breaking, known as Hilbert space fragmentation. This phenomenon, in which the Hilbert space of the system becomes split into dynamically disconnected sectors, can lead to non-thermal states and persistent oscillations, both in integrable and non-integrable settings. Together, these studies presented in the thesis advance the theoretical understanding of the non-equilibrium dynamics of integrable quantum models.

# Goals

The central goal of the thesis is to further the understanding of integrable quantum systems in both in- and out-of-equilibrium settings. This aim was pursued in two main research directions: one of my objectives was to strengthen the theoretical foundations of GHD, by providing a mathematically rigorous proof of one of its core assumptions regarding the mean values of the current operators. To this end, first I studied the current mean values in the XXZ model in finite volume and proved a mean value formula that supports the assumptions of GHD, and that can be generalized to other integrable lattice systems with a $U(1)$ symmetry. To further extend this theoretical foundation, I also studied the XYZ model and was able to prove a similar formula for the expectation values of currents.

The other main line of research was the less well-defined search for „simple" integrable systems. The goal behind this aspect of my work was similarly to better establish the applicability of GHD. Since GHD is a hydrodynamic description that relies on assumptions, it would be reassuring from a theoretical point of view to compare and validate its predictions against results obtained from the microscopic laws governing integrable systems. This is, however, impossible in most models. Because of this, it is useful to find such integrable systems that are „simpler" in a sense, and that might provide a setting to verify GHD. In my thesis, I did not attempt this verification but studied two such simple candidate models that are also interesting in themselves. Particularly, one of these systems, called the folded XXZ model exhibits Hilbert space fragmentation, which is a distinct way of thermalization breaking compared to integrability.

# Methods

One of the most exciting properties of integrable models is that they can be solved exactly with analytical tools. Accordingly, in most of the dissertation, I used analytic techniques, such as the coordinate and algebraic Bethe Ansatz (CBA and ABA, respectively). These powerful tools allow one to obtain the eigenstates and eigenvalues of integrable Hamiltonians and also to compute more involved quantities, such as the mean values of local observables and certain correlation functions. In addition to the well-known techniques of CBA and ABA, I also employed the more exotic generalized ABA to treat the XYZ model.

Despite the exact solvability of integrable systems, there are situations where one needs to resort to numerical techniques. In my dissertation, I used two types of numerical methods: to calculate the expectation values of observables in finite volumes, I utilized exact diagonalization (ED), which means the explicit construction and diagonalization of the Hamiltonian matrix. ED was used mostly to verify analytic results concerning the mean values of local operators and to calculate the level spacing statistics of the folded XXZ model and its non-integrable extension.

The second, more sophisticated numerical technique that I used is the so-called infinite Time-Evolving Block Decimation (iTEBD) algorithm. This is a powerful numerical method that is based on the Matrix Product State (MPS) representation of the wave function of the system in question. It allows one to approximate the time evolution of models directly in the thermodynamic limit by exploiting translational invariance. In my work, I used iTEBD to simulate time evolution in different medium-range spin-chain models following global quantum quenches. My implementation of the algorithm is based on a previously existing code that I modified to incorporate longer than nearest-neighbor interactions.

# Original results

1. I worked out a rigorous proof for a formula that describes the mean values of the (generalized) current operators of the XXZ spin-$1/2$ chain in finite volume. (The formula was originally conjectured by my supervisor.) The proof is based on a form-factor expansion that I established for the XXZ model by using its ABA solution and the solution of the quantum inverse scattering problem. Using this expansion theorem, I first calculated the form factors of the charge operators, and then, by utilizing the continuity equation, I computed the form factors of the currents. By the repeated application of the form-factor expansion theorem and with the help of the newly obtained current form factors, I calculated the expectation values of the (generalized) current operators in arbitrary eigenstates of the XXZ model in finite volume. This result provides a rigorous theoretical foundation for one of the underlying assumptions of GHD and was published in [1].

2. I extended the results described in the previous point to the non $U(1)$ symmetric XYZ spin-$1/2$ model. To this end, I used the known generalized ABA solution of the XYZ model and the algebraic construction of the current operators that was first found by my supervisor. Using these tools, I calculated the mean values of the (generalized) current operators in arbitrary eigenstates of the XYZ model in finite volume and proved a similar formula to that of the previous point. The results were published in [2].

3. As part of a larger collaboration, I re-derived and studied the so-called folded XXZ model, which is an integrable system with a particularly simple two-particle scattering phase. Our results were published in [3]. My main contributions to [3] were the calculation of the charges of the folded model starting from the known charges of the XXZ spin-$1/2$ chain, the calculation of the ground state properties of the folded model and the numerical simulation of the time-evolution, following a quantum quench. Additionally, I also contributed to obtaining the Bethe Ansatz solution of the folded model and to the calculation of the exact time evolution of the so-called emptiness formation probability, following a specific quantum quench scenario.

   The folded XXZ model exhibits Hilbert space fragmentation, leading to the breakdown of thermalization in this system. I studied this phenomenon by numerically simulating the real time-evolution following a global quantum quench using the iTEBD algorithm. My implementation of the numerical method is based on previously existing code that I modified to the needs of the medium-range folded XXZ model. My numerical results support the theoretical considerations and show persistent oscillations that signal the breakdown of equilibration. I also performed the same analysis in the case of a non-integrable extension of the folded XXZ model, whose non-integrability I demonstrated by numerically calculating its level spacing statistics. The results related to this non-integrable extension were published in [4].

4. As part of a collaboration, I introduced and studied a one-dimensional anyon-like spin chain, which constitutes one of the simplest interacting integrable systems. My main contribution to this research was the derivation of the charges of the model through its connection to a particular limit of the integrable Trotterization of the XXZ spin chain. Moreover, I also contributed to solving the model via CBA and to the analytical and numerical calculations of the entanglement entropy in the system, in both in- and out-of-equilibrium situations. Our results were published in [5].

# Publications

[1] M. Borsi, B. Pozsgay, and L. Pristyák, “Current Operators in Bethe Ansatz and Generalized Hydrodynamics: An Exact Quantum-Classical Correspondence,” *Phys. Rev. X* **10** (Mar, 2020) 011054. `https://link.aps.org/doi/10.1103/PhysRevX.10.011054`.

[2] L. Pristyák and B. Pozsgay, “Current mean values in the XYZ model,” *SciPost Phys.* **14** (Jun, 2023) 158. `https://scipost.org/10.21468/SciPostPhys.14.6.158`.

[3] B. Pozsgay, T. Gombor, A. Hutsalyuk, Y. Jiang, L. Pristyák, and E. Vernier, “Integrable spin chain with Hilbert space fragmentation and solvable real-time dynamics,” *Phys. Rev. E* **104** (Oct, 2021) 044106. `https://link.aps.org/doi/10.1103/PhysRevE.104.044106`.

[4] M. Borsi, L. Pristyák, and B. Pozsgay, “Matrix Product Symmetries and Breakdown of Thermalization from Hard Rod Deformations,” *Phys. Rev. Lett.* **131** (Jul, 2023) 037101. `https://link.aps.org/doi/10.1103/PhysRevLett.131.037101`.

[5] B. Pozsgay, A. Hutsalyuk, L. Pristyák, and G. Takács, “Sublattice entanglement in an exactly solvable anyonlike spin ladder,” *Phys. Rev. E* **106** (Oct, 2022) 044120. `https://link.aps.org/doi/10.1103/PhysRevE.106.044120`.

# Further publications not connected to the thesis statements

[P1] O. Pomponio, L. Pristyák, and G. Takács, “Quasi-particle spectrum and entanglement generation after a quench in the quantum Potts spin chain,” *Journal of Statistical Mechanics: Theory and Experiment* **2019** no. 1, (Jan, 2019) 013104. `https://dx.doi.org/10.1088/1742-5468/aafa80`.

[P2] A. Hutsalyuk, B. Pozsgay, and L. Pristyák, “The LeClair-Mussardo series and nested Bethe Ansatz,” *Nuclear Physics B* **964** (Mar, 2021) 115306. `https://www.sciencedirect.com/science/article/pii/S0550321321000031`.

[P3] M. Borsi, B. Pozsgay, and L. Pristyák, “Current operators in integrable models: a review,” *Journal of Statistical Mechanics: Theory and Experiment* **2021** no. 9, (Sep, 2021) 094001. `https://dx.doi.org/10.1088/1742-5468/ac0f6b`.